

# A criterion for coincidence of the entanglement-assisted classical capacity and the Holevo capacity of a quantum channel.\*

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## Abstract

It is easy to show coincidence of the entanglement-assisted classical capacity and the Holevo capacity for any c-q channel between finite dimensional quantum systems. In this paper we prove the converse assertion: coincidence of the above-mentioned capacities of a quantum channel implies that the  $\chi$ -essential part of this channel is a c-q channel (the  $\chi$ -essential part is a restriction of a channel obtained by discarding all input states useless for transmission of classical information).

The above observations are generalized to infinite dimensional quantum channels with linear constraints by using the obtained conditions for coincidence of the quantum mutual information and the constrained Holevo capacity.

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# 1 Introduction

In this paper we specify the necessary and sufficient conditions for coincidence of the Holevo capacity  $\bar{C}(\Phi)$  and the entanglement-assisted (classical) capacity  $C_{\text{ea}}(\Phi)$  of a quantum channel  $\Phi$  presented in [16] and generalize these conditions to the infinite dimensional case by using the results concerning reversibility of a quantum channel with respect to families of pure states obtained in [17].

In contrast to an intuitive point of view, the class of channels  $\Phi$  such that

$$\bar{C}(\Phi) = C_{\text{ea}}(\Phi) \tag{1}$$

does not coincide with the class of entanglement-breaking channels. The examples of entanglement-breaking channels, in particular, of q-c channels, for which  $\bar{C}(\Phi) < C_{\text{ea}}(\Phi)$ , can be found in [1, 6, 16]. In [2] the non-entanglement-breaking channel such that equality (1) holds is constructed. A criterion of this equality for the class of q-c channels defined by quantum observables is recently obtained in [6].

In the first part of the paper we show that equality (1) holds for a finite dimensional quantum channel  $\Phi$  if (correspondingly, only if) this channel (correspondingly, the  $\chi$ -essential part of this channel) belongs to the class of c-q channels (the  $\chi$ -essential part is defined as a restriction of a channel to the set of states supported by the minimal subspace containing elements of *all* ensembles optimal for this channel in the sense of the Holevo capacity, see Definition 1). Thus, assuming that we deal with channels reduced to their  $\chi$ -essential parts, we may say that the class of channels, for which equality (1) holds, coincides with the class of c-q channels.

In the second part of the paper the above observations are generalized to infinite dimensional channels with linear constraints by studying the equality conditions for the general inequality

$$\bar{C}(\Phi, \rho) \leq I(\Phi, \rho), \tag{2}$$

connecting the constrained Holevo capacity  $\bar{C}(\Phi, \rho)$  and the quantum mutual information  $I(\Phi, \rho)$  of a quantum channel  $\Phi$  at a state  $\rho$ . It is shown that the equality in (2) implies that the restriction of the channel  $\Phi$  to the set of states supported by the subspace  $\text{supp } \rho$  (the support of  $\rho$ ) is a c-q channel. The converse implication holds if the state  $\rho$  is diagonalizable in the basis from the c-q representation of the above restriction of the channel  $\Phi$ .

## 2 Preliminaries

Let  $\mathcal{H}$  be either a finite dimensional or separable Hilbert space,  $\mathfrak{B}(\mathcal{H})$  and  $\mathfrak{T}(\mathcal{H})$  – the Banach spaces of all bounded operators in  $\mathcal{H}$  and of all trace-class operators in  $\mathcal{H}$  correspondingly,  $\mathfrak{S}(\mathcal{H})$  – the closed convex subset of  $\mathfrak{T}(\mathcal{H})$  consisting of positive operators with unit trace called *states* [12].

Denote by  $I_{\mathcal{H}}$  and  $\text{Id}_{\mathcal{H}}$  the unit operator in a Hilbert space  $\mathcal{H}$  and the identity transformation of the Banach space  $\mathfrak{T}(\mathcal{H})$  correspondingly.

A linear completely positive trace preserving map  $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$  is called *quantum channel* [12]. The Stinespring theorem implies existence of a Hilbert space  $\mathcal{H}_E$  and of an isometry  $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$  such that

$$\Phi(A) = \text{Tr}_{\mathcal{H}_E} V A V^*, \quad A \in \mathfrak{T}(\mathcal{H}_A). \quad (3)$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni A \mapsto \hat{\Phi}(A) = \text{Tr}_{\mathcal{H}_B} V A V^* \in \mathfrak{T}(\mathcal{H}_E) \quad (4)$$

is called *complementary* to the channel  $\Phi$  [7].<sup>1</sup> The complementary channel is defined uniquely (cf.[7, the Appendix]): if  $\hat{\Phi}' : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{E'})$  is a channel defined by (4) via the Stinespring isometry  $V' : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E'}$  then the channels  $\hat{\Phi}$  and  $\hat{\Phi}'$  are *isometrically equivalent* in the sense that there is a partial isometry  $W : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}$  such that

$$\hat{\Phi}'(A) = W \hat{\Phi}(A) W^*, \quad \hat{\Phi}(A) = W^* \hat{\Phi}'(A) W, \quad A \in \mathfrak{T}(\mathcal{H}_A). \quad (5)$$

A channel  $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$  is called *classical-quantum* (briefly a *c-q channel*) if it has the following representation

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k | \rho | k \rangle \sigma_k, \quad (6)$$

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<sup>1</sup>The quantum channel  $\hat{\Phi}$  is also called *conjugate* to the channel  $\Phi$  [10].

where  $\{|k\rangle\}$  is an orthonormal basis in  $\mathcal{H}_A$  and  $\{\sigma_k\}$  is a collection of states in  $\mathfrak{S}(\mathcal{H}_B)$  [12].

Let  $H(\rho)$  and  $H(\rho\|\sigma)$  be respectively the von Neumann entropy of the state  $\rho$  and the quantum relative entropy of the states  $\rho$  and  $\sigma$  [11, 12]. Let

$$\chi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\rho_i\|\bar{\rho}) = H(\bar{\rho}) - \sum_i \pi_i H(\rho_i)$$

be the  $\chi$ -quantity of an ensemble  $\{\pi_i, \rho_i\}$  of quantum states with the average state  $\bar{\rho}$ , where the second expression is valid under the condition  $H(\bar{\rho}) < +\infty$ .

The constrained Holevo capacity of the channel  $\Phi$  at a state  $\rho \in \mathfrak{S}(\mathcal{H}_A)$  is defined as follows

$$\bar{C}(\Phi, \rho) \doteq \sup_{\{\pi_i, \rho_i\}, \bar{\rho}=\rho} \chi(\{\pi_i, \Phi(\rho_i)\}) \quad (7)$$

(the supremum is over all ensembles of states in  $\mathfrak{S}(\mathcal{H}_A)$  with the average state  $\rho$ ). If  $H(\Phi(\rho)) < +\infty$  then

$$\bar{C}(\Phi, \rho) = H(\Phi(\rho)) - \hat{H}_\Phi(\rho), \quad (8)$$

where  $\hat{H}_\Phi(\rho) = \inf_{\{\pi_i, \rho_i\}, \bar{\rho}=\rho} \sum_i \pi_i H(\Phi(\rho_i))$  (the infimum here can be taken over ensembles of pure states by concavity of the function  $\rho \mapsto H(\Phi(\rho))$ ).

In finite dimensions the quantum mutual information of the channel  $\Phi$  at a state  $\rho \in \mathfrak{S}(\mathcal{H}_A)$  is defined as follows (cf.[12])

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\hat{\Phi}(\rho)). \quad (9)$$

Since in infinite dimensions the terms in the right side of (9) may be infinite, it is reasonable to define the quantum mutual information by the following formula

$$I(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(|\varphi_\rho\rangle\langle\varphi_\rho|) \parallel \Phi(\rho) \otimes \varrho),$$

where  $\varphi_\rho$  is a purification vector<sup>2</sup> in  $\mathcal{H}_A \otimes \mathcal{H}_R$  for the state  $\rho \in \mathfrak{S}(\mathcal{H}_A)$  and  $\varrho = \text{Tr}_{\mathcal{H}_A} |\varphi_\rho\rangle\langle\varphi_\rho|$  is a state in  $\mathfrak{S}(\mathcal{H}_R)$  isomorphic to  $\rho$ . If  $H(\rho)$  and  $H(\Phi(\rho))$  are finite then the last formula for  $I(\Phi, \rho)$  coincides with (9).

If  $\rho$  is a state with finite entropy then the quantum mutual information of an arbitrary channel  $\Phi$  at the state  $\rho$  can be expressed as follows

$$I(\Phi, \rho) = H(\rho) + \bar{C}(\Phi, \rho) - \bar{C}(\hat{\Phi}, \rho) = \bar{C}(\Phi, \rho) + \Delta_\Phi(\rho), \quad (10)$$

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<sup>2</sup>This means that  $\text{Tr}_{\mathcal{H}_R} |\varphi_\rho\rangle\langle\varphi_\rho| = \rho$ .

where  $\Delta_\Phi(\rho) = H(\rho) - \bar{C}(\hat{\Phi}, \rho) \geq 0$  (by monotonicity of the relative entropy). If  $H(\Phi(\rho))$  and  $H(\hat{\Phi}(\rho))$  are finite then expression (10) can be easily derived from (8) and (9), since  $\hat{H}_\Phi \equiv \hat{H}_{\hat{\Phi}}$  (this follows from coincidence of the functions  $\rho \mapsto H(\Phi(\rho))$  and  $\rho \mapsto H(\hat{\Phi}(\rho))$  on the set of pure states), in general case it can be proved by using Lemma 1 in the Appendix.

### 3 Finite dimensional channels

The Holevo capacity of a finite dimensional channel  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  is defined as follows

$$\bar{C}(\Phi) \doteq \max_{\{\pi_i, \rho_i\}} \chi(\{\pi_i, \Phi(\rho_i)\}) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \bar{C}(\Phi, \rho), \quad (11)$$

where the first maximum is over all ensembles of states in  $\mathfrak{S}(\mathcal{H}_A)$ .

An ensemble  $\{\pi_i, \rho_i\}$ , at which the first maximum in (11) is attained, is called *optimal* for the channel  $\Phi$  [14].

By the HSW theorem the classical capacity of the channel  $\Phi$  can be expressed by the following regularization formula

$$C(\Phi) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}).$$

By the BSST theorem the entanglement-assisted classical capacity of the channel  $\Phi$  is determined as follows

$$C_{\text{ea}}(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I(\Phi, \rho). \quad (12)$$

According to [16] introduce the following notion.

**Definition 1.** Let  $\mathcal{H}_\Phi^\chi$  be the minimal subspace of  $\mathcal{H}_A$  containing elements of all optimal ensembles for the channel  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ . The restriction  $\Phi_\chi$  of the channel  $\Phi$  to the set  $\mathfrak{S}(\mathcal{H}_\Phi^\chi)$  is called the  $\chi$ -essential part (subchannel) of the channel  $\Phi$ .

It is clear that  $\mathcal{H}_\Phi^\chi = \mathcal{H}_A$  means existence of an optimal ensemble for the channel  $\Phi$  with the full rank average state. If  $\mathcal{H}_\Phi^\chi \neq \mathcal{H}_A$  then pure states corresponding to vectors in  $\mathcal{H}_A \setminus \mathcal{H}_\Phi^\chi$  are useless for non-entangled coding of classical information and hence it is natural to consider the  $\chi$ -essential

subchannel  $\Phi_\chi$  instead of the channel  $\Phi$  dealing with the Holevo capacity of the channel  $\Phi$  (which coincides with the classical capacity if  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ ).

By definition  $\bar{C}(\Phi_\chi) = \bar{C}(\Phi)$ . Hence  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$  implies that  $C_{\text{ea}}(\Phi_\chi) = C_{\text{ea}}(\Phi)$ . Thus, in this case speaking about the entanglement-assisted capacity of the channel  $\Phi$  we may also consider the  $\chi$ -essential subchannel  $\Phi_\chi$  instead of the channel  $\Phi$ .

Now we can formulate our main result.

**Theorem 1.** *Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a quantum channel.*

A) *If  $\Phi$  is a c-q channel then  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ ; <sup>3</sup>*

B) *If  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$  then the  $\chi$ -essential part of  $\Phi$  is a c-q channel.*

**Remark 1.** The presence of "the  $\chi$ -essential part of  $\Phi$ " in assertion B is natural, since the remarks before Theorem 1 show that the equality  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$  can not give information about action of the channel  $\Phi$  on states not supported by the subspace  $\mathcal{H}_\Phi^\chi$ . This conclusion is confirmed by the example of non-entanglement-breaking channel  $\Phi$  such that  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$  proposed in [2] and mathematically described in [16, Example 3].

**Corollary 1.** *Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a quantum channel for which there exists an optimal ensemble with the full rank average state. Then*

$$C_{\text{ea}}(\Phi) = \bar{C}(\Phi) \quad \Leftrightarrow \quad \Phi \text{ is a c-q channel.}$$

**Proof of Theorem 1.** A) If the channel  $\Phi$  has representation (6) then  $\Phi = \Phi \circ \Pi$ , where  $\Pi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k | \rho | k \rangle |k\rangle \langle k|$ . By the chain rule for the quantum mutual information we have

$$I(\Phi, \rho) = I(\Phi \circ \Pi, \rho) \leq I(\Phi, \Pi(\rho)).$$

Hence the supremum in expression (12) can be taken only over states diagonalizable in the basis  $\{|k\rangle\}$ . So, to prove the equality  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$  it suffices to show that  $I(\Phi, \rho) = \bar{C}(\Phi, \rho)$  for any such state  $\rho$ .

Since the channel  $\Phi$  has representation (6), Proposition 1 in [17] implies that the channel  $\hat{\Phi}$  is reversible with respect to the family  $\{|k\rangle \langle k|\}$  of pure states (i.e. there exists a channel  $\Psi$  such that  $\Psi(\hat{\Phi}(|k\rangle \langle k|)) = |k\rangle \langle k|$  for all  $k$ ). It follows that  $\bar{C}(\hat{\Phi}, \rho) = H(\rho)$  for any state  $\rho$  diagonalizable in the

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<sup>3</sup>This assertion seems well known. I would be grateful for the reference.

basis  $\{|k\rangle\}$ . By expression (10) the last equality is equivalent to the equality  $I(\Phi, \rho) = \bar{C}(\Phi, \rho)$ .

B) By replacing the channel  $\Phi$  by its  $\chi$ -essential subchannel we may assume that there exists an optimal ensemble  $\{\pi_i, \rho_i\}$  of pure states for the channel  $\Phi$  with the full rank average  $\bar{\rho}$ . By expression (10) we have

$$C_{\text{ea}}(\Phi) = \bar{C}(\Phi) \Rightarrow I(\Phi, \bar{\rho}) = \bar{C}(\Phi, \bar{\rho}) \Leftrightarrow \bar{C}(\hat{\Phi}, \bar{\rho}) = H(\bar{\rho}). \quad (13)$$

Since  $H(\bar{\rho}) = \chi(\{\pi_i, \rho_i\})$  and  $\bar{C}(\hat{\Phi}, \bar{\rho}) = \chi(\{\pi_i, \hat{\Phi}(\rho_i)\})$  (this can be shown by using (8) and coincidence of the functions  $\hat{H}_\Phi$  and  $\hat{H}_{\hat{\Phi}}$ ), the last equality in (13) and Theorem 5 in [17] show that  $\Phi$  is a c-q channel (it coincides with the complementary channel to the channel  $\hat{\Phi}$  up to isometrical equivalence).  $\square$

Theorem 1 makes it possible to strengthen Proposition 4 in [16] as follows.

**Proposition 1.** *Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a degradable channel. Then the following statements are equivalent:*

- (i)  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ ;
- (ii)  $C_{\text{ea}}(\Phi) = \bar{C}(\Phi) = \log \dim \mathcal{H}_A$ ;
- (iii)  $\Phi$  is a c-q channel having representation (6) in which  $\{\sigma_k\}$  is a collection of states in  $\mathcal{H}_B$  with pairwise orthogonal supports.

**Proof.** (i)  $\Rightarrow$  (ii) follows from Proposition 4 in [16].

(ii)  $\Rightarrow$  (iii).  $\bar{C}(\Phi) = \log \dim \mathcal{H}_A$  implies  $\Phi_\chi = \Phi$  and hence Theorem 1B shows that  $\Phi$  is a c-q channel.

Assume that the states of the collection  $\{\sigma_k\}$  from representation (6) of the channel  $\Phi$  are decomposed as follows  $\sigma_k = \sum_{i=1}^{\dim \mathcal{H}_B} |\psi_{ki}\rangle\langle\psi_{ki}|$ . Then  $\Phi(\rho) = \sum_{k,i} W_{ki} \rho W_{ki}^*$ , where  $W_{ki} = |\psi_{ki}\rangle\langle k|$  ( $\{|k\rangle\}$  is the basis from representation (6)), and by using the standard representation of the complementary channel (see [7, formula (11)]) we obtain

$$\hat{\Phi}(\rho) = \sum_{k,l=1}^{\dim \mathcal{H}_A} \langle k|\rho|l\rangle |k\rangle\langle l| \otimes \sum_{i,j=1}^{\dim \mathcal{H}_B} \langle \psi_{lj}|\psi_{ki}\rangle |i\rangle\langle j| \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B). \quad (14)$$

Since  $\Phi$  is a degradable channel having representation (6), we have  $\hat{\Phi}(|k\rangle\langle l|) = \Psi \circ \Phi(|k\rangle\langle l|) = 0$  for all  $k \neq l$ . Hence (14) shows that  $\langle \psi_{lj}|\psi_{ki}\rangle = 0$  for all  $i, j$  and all  $k \neq l$ , which means that  $\text{supp} \sigma_k \perp \text{supp} \sigma_l$  for all  $k \neq l$ .

(iii)  $\Rightarrow$  (i) follows from Theorem 1A.  $\square$

## 4 Infinite dimensional channels

### 4.1 On coincidence of the constrained Holevo capacity and the quantum mutual information

Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be an arbitrary finite or infinite dimensional quantum channel. In this subsection we consider conditions for the equality in the general inequality

$$\bar{C}(\Phi, \rho) \leq I(\Phi, \rho), \quad \rho \in \mathfrak{S}(\mathcal{H}_A)$$

This inequality can be proved by using expression (10) valid under the condition  $H(\rho) < +\infty$  and a simple approximation.

We have to introduce some additional notions.

A continuous (generalized) ensemble of quantum states can be defined as a Borel probability measure  $\mu$  on the set  $\mathfrak{S}(\mathcal{H})$ . The  $\chi$ -quantity of such ensemble (measure)  $\mu$  is defined as follows (cf. [8])

$$\chi(\mu) = \int_{\mathfrak{S}(\mathcal{H})} H(\rho \| \bar{\rho}(\mu)) \mu(d\rho), \quad (15)$$

where  $\bar{\rho}(\mu)$  is the barycenter of the measure  $\mu$  defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho).$$

If  $H(\bar{\rho}(\mu)) < +\infty$  then  $\chi(\mu) = H(\bar{\rho}(\mu)) - \int_{\mathfrak{S}(\mathcal{H})} H(\rho) \mu(d\rho)$  [8].

Denote by  $\mathcal{P}(\mathcal{A})$  the set of all Borel probability measures on a closed subset  $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$  endowed with the weak convergence topology [13].

The image of a continuous ensemble  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  under a channel  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  is a continuous ensemble corresponding to the measure  $\Phi(\mu) \doteq \mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_B))$ . Its  $\chi$ -quantity can be expressed as follows

$$\begin{aligned} \chi(\Phi(\mu)) &\doteq \int_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho) \| \Phi(\bar{\rho}(\mu))) \mu(d\rho) \\ &= H(\Phi(\bar{\rho}(\mu))) - \int_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)) \mu(d\rho), \end{aligned} \quad (16)$$

where the second formula is valid under the condition  $H(\Phi(\bar{\rho}(\mu))) < +\infty$ .



The constrained Holevo capacity defined by (7) can be also expressed as follows

$$\bar{C}(\Phi, \rho) = \sup_{\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A)), \bar{\rho}(\mu) = \rho} \chi(\Phi(\mu)). \quad (17)$$

This expression follows from Corollary 1 in [8] with  $\mathcal{A} = \{\rho\}$  (where the constrained Holevo capacity is denoted  $\chi_\Phi(\rho)$ ).

**Theorem 2.** *Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a quantum channel and  $\rho$  be a state in  $\mathfrak{S}(\mathcal{H}_A)$  with the support  $\mathcal{H}_\rho$ .*

A) *If the restriction  $\Phi|_{\mathfrak{S}(\mathcal{H}_\rho)}$  of the channel  $\Phi$  to the set  $\mathfrak{S}(\mathcal{H}_\rho)$  is a c-q channel having representation (6) with  $\mathcal{H}_A = \mathcal{H}_\rho$ , in which  $\{|k\rangle\}$  is an orthonormal basis of eigenvectors of the state  $\rho$ , then  $\bar{C}(\Phi, \rho) = I(\Phi, \rho) \leq +\infty$ .*

B) *If  $H(\rho) < +\infty$  and the following condition holds*

$$\exists \mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A)) \text{ such that } \bar{\rho}(\mu) = \rho \text{ and } \bar{C}(\Phi, \rho) = \chi(\Phi(\mu)), \quad (18)$$

*which means that the supremum in (17) is attainable, then*

$$\bar{C}(\Phi, \rho) = I(\Phi, \rho) < +\infty \quad \Rightarrow \quad \Phi|_{\mathfrak{S}(\mathcal{H}_\rho)} \text{ is a c-q channel.}$$

*Condition (18) is valid if either  $H(\Phi(\rho)) < +\infty$  or one of the functions  $\sigma \mapsto H(\Phi(\sigma) \parallel \Phi(\rho))$ ,  $\sigma \mapsto H(\hat{\Phi}(\sigma) \parallel \hat{\Phi}(\rho))$  is continuous and bounded on the set  $\text{extr}\mathfrak{S}(\mathcal{H}_A)$ .*

**Remark 2.** Example 5 in [16] shows that the assertion of Theorem 2A is not valid if the state  $\rho$  is not diagonalizable in the basis  $\{|k\rangle\}$ .

**Proof.** A) We may assume that  $\mathcal{H}_\rho = \mathcal{H}_A$ . Let  $\rho = \sum_{k=1}^{\dim \mathcal{H}_A} \lambda_k |k\rangle\langle k|$  and  $\rho_n = [\sum_{k=1}^n \lambda_k]^{-1} \sum_{k=1}^n \lambda_k |k\rangle\langle k|$ . Then the sequence  $\{\rho_n\}$  converges to the state  $\rho$  and  $H(\rho_n) < +\infty$  for all  $n$ .

If the channel  $\Phi$  has representation (6) then the channel  $\hat{\Phi}$  is reversible with respect to the family  $\{|k\rangle\langle k|\}_{k=1}^{\dim \mathcal{H}_A}$  by Proposition 1 in [17]. Hence  $\bar{C}(\hat{\Phi}, \rho_n) = H(\rho_n)$  and expression (10) implies  $\bar{C}(\Phi, \rho_n) = I(\Phi, \rho_n)$  for all  $n$ . It follows that  $\bar{C}(\Phi, \rho) = I(\Phi, \rho)$ , since by using concavity and lower semicontinuity of the nonnegative functions  $\rho \mapsto \bar{C}(\Phi, \rho)$  and  $\rho \mapsto I(\Phi, \rho)$  it is easy to show that

$$\lim_{n \rightarrow +\infty} \bar{C}(\Phi, \rho_n) = \bar{C}(\Phi, \rho) \leq +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} I(\Phi, \rho_n) = I(\Phi, \rho) \leq +\infty.$$

B) Without loss of generality we may consider that the measure  $\mu$  in (18) belongs to the set  $\mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A))$ . This follows from convexity of the function  $\sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho))$ , since for an arbitrary measure  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  there exists a measure  $\hat{\mu} \in \mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A))$  such that  $\bar{\rho}(\hat{\mu}) = \bar{\rho}(\mu)$  and  $\int f(\sigma)\hat{\mu}(d\sigma) \geq \int f(\sigma)\mu(d\sigma)$  for any convex lower semicontinuous nonnegative function  $f$  on the set  $\mathfrak{S}(\mathcal{H}_A)$  (this measure  $\hat{\mu}$  can be constructed by using the arguments from the proof of the Theorem in [8]).

By expression (10) the equality  $\bar{C}(\Phi, \rho) = I(\Phi, \rho)$  is equivalent to the equality  $H(\rho) = \bar{C}(\hat{\Phi}, \rho)$ . By the remark after Lemma 1 in the Appendix condition (18) implies that  $\bar{C}(\hat{\Phi}, \rho) = \chi(\hat{\Phi}(\mu))$ . Since  $H(\rho) = \chi(\mu)$ , the equality  $H(\rho) = \bar{C}(\hat{\Phi}, \rho)$  shows that the channel  $\hat{\Phi}$  preserves the  $\chi$ -quantity of the measure  $\mu$ . By Theorem 5 in [17] the restriction to the set  $\mathfrak{S}(\mathcal{H}_\rho)$  of the complementary channel to the channel  $\hat{\Phi}$  is a c-q channel.

If  $H(\Phi(\rho)) < +\infty$  then condition (18) holds by Corollary 2 in [8].

If the function  $\sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho))$  is continuous and bounded on the set  $\text{extr}\mathfrak{S}(\mathcal{H}_A)$  then the function  $\mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A)) \ni \mu \mapsto \chi(\Phi(\mu))$  is continuous by the definition of the weak convergence. Since the subset of  $\mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A))$  consisting of measures with the barycenter  $\rho$  is compact by Proposition 2 in [8], the last function attains its least upper bound on this subset.

If the function  $\sigma \mapsto H(\hat{\Phi}(\sigma)\|\hat{\Phi}(\rho))$  is continuous and bounded on the set  $\text{extr}\mathfrak{S}(\mathcal{H}_A)$  then the similar arguments show attainability of the supremum in the definition of the value  $\bar{C}(\hat{\Phi}, \rho)$ , which is equivalent to (18) by the remark after Lemma 1 in the Appendix.  $\square$

## 4.2 Conditions for the equality $C_{\text{ea}}(\Phi) = \bar{C}(\Phi)$ for channels with linear constraints

In this subsection we derive from Theorem 2 the necessary and sufficient conditions for coincidence of the entanglement-assisted classical capacity and the Holevo capacity of an infinite dimensional channel  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  with the constraint defined by the inequality

$$\text{Tr} H \rho \leq h, \quad h > 0, \quad (19)$$

where  $H$  is a positive operator in  $\mathcal{H}_A$  – Hamiltonian of the input quantum system.<sup>4</sup> The operational definitions of the unassisted and the entanglement-

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<sup>4</sup>Speaking about capacities of infinite dimensional quantum channels we have to impose particular constraints on the choice of input code-states to avoid infinite values of

assisted classical capacities of a quantum channel with constraint (19) are given in [5], where the corresponding generalizations of the HSW and BSST theorems are proved.

The Holevo capacity of the channel  $\Phi$  with constraint (19) can be defined as follows

$$\bar{C}(\Phi|H, h) = \sup_{\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A)), \text{Tr} H \bar{\rho}(\mu) \leq h} \chi(\Phi(\mu)), \quad (20)$$

where the supremum can be taken over all measures in  $\mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  supported by pure states or, equivalently,

$$\bar{C}(\Phi|H, h) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A), \text{Tr} H \rho \leq h} \bar{C}(\Phi, \rho). \quad (21)$$

If the supremum in (20) is achieved at a measure  $\mu_*$  then this measure is called optimal for the channel  $\Phi$  with constraint (19). The sufficient condition for existence of optimal measures and examples showing that optimal measures do not exist in general can be found in [8].

An optimal measure for the channel  $\Phi$  with constraint (19) exists if and only if the supremum in (21) is achieved at a state  $\rho$  for which the condition (18) holds (the sufficient conditions for this are presented at the end of Theorem 2).

By the generalized HSW theorem ([5, Proposition 3]) the classical capacity of the channel  $\Phi$  with constraint (19) can be expressed by the following regularization formula

$$C(\Phi|H, h) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}|H_n, nh),$$

where  $H_n = H \otimes I \otimes \dots \otimes I + I \otimes H \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes H$  (each of  $n$  summands consists of  $n$  multiples).

By the generalized BSST theorem ([5, Proposition 4]) the entanglement-assisted classical capacity of the channel  $\Phi$  with constraint (19) is determined as follows

$$C_{\text{ea}}(\Phi|H, h) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A), \text{Tr} H \rho \leq h} I(\Phi, \rho). \quad (22)$$

This expression is proved in [5] under the particular technical conditions on the channel  $\Phi$  and the operator  $H$ , which can be removed by using the approximation method [9]. We will assume that expression (22) is valid.

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the capacities and to be consistent with the physical implementation of the process of information transmission [5].

Theorem 2 implies the following conditions for coincidence of  $\bar{C}(\Phi|H, h)$  and  $C_{\text{ea}}(\Phi|H, h)$ .

**Corollary 2.** *Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a quantum channel and  $H$  be a positive operator in  $\mathfrak{S}(\mathcal{H}_A)$ .*

A) *If  $\Phi$  is a c-q channel having representation (6) and the operator  $H$  is diagonalizable<sup>5</sup> in the basis  $\{|k\rangle\}$  in (6) then  $\bar{C}(\Phi|H, h) = C_{\text{ea}}(\Phi|H, h)$ ;*

B) *If  $\bar{C}(\Phi|H, h) = C_{\text{ea}}(\Phi|H, h) < +\infty$  and the supremum in (20) is achieved at a measure  $\mu_*$  such that  $H(\bar{\rho}(\mu_*)) < +\infty$  then the restriction of the channel  $\Phi$  to the set  $\mathfrak{S}(\mathcal{H}_{\bar{\rho}(\mu_*)})$ ,  $\mathcal{H}_{\bar{\rho}(\mu_*)} = \text{supp } \bar{\rho}(\mu_*)$ , is a c-q channel.*

*The condition concerning existence of the measure  $\mu_*$  holds if the subset of  $\mathfrak{S}(\mathcal{H}_A)$  defined by inequality (19) is compact and the output entropy of the channel  $\Phi$  (the function  $\rho \mapsto H(\Phi(\rho))$ ) is continuous on this subset.*

**Remark 3.** Example 5 in [16] shows that the assertion of Corollary 2A is not valid if the operator  $H$  is not diagonalizable in the basis  $\{|k\rangle\}$ .

**Proof.** A) If the channel  $\Phi$  has representation (6) then  $\Phi = \Phi \circ \Pi$ , where  $\Pi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle |k\rangle\langle k|$ . By the chain rule for the quantum mutual information we have

$$I(\Phi, \rho) = I(\Phi \circ \Pi, \rho) \leq I(\Phi, \Pi(\rho)). \quad (23)$$

If the operator  $H$  is diagonalizable in the basis  $\{|k\rangle\}$  then the inequality  $\text{Tr} H \rho \leq h$  implies the inequality  $\text{Tr} H \Pi(\rho) \leq h$ . Hence (23) shows that the supremum in expression (22) can be taken only over states diagonalizable in the basis  $\{|k\rangle\}$ . Since  $\bar{C}(\Phi, \rho) = I(\Phi, \rho)$  for any such state  $\rho$  by Theorem 2A, we have  $\bar{C}(\Phi|H, h) = C_{\text{ea}}(\Phi|H, h)$ .

B) The main assertion immediately follows from Theorem 2A, since its condition implies  $\bar{C}(\Phi, \bar{\rho}(\mu_*)) = I(\Phi, \bar{\rho}(\mu_*))$ . The sufficient condition for existence of the measure  $\mu_*$  follows from the Theorem in [8].  $\square$

**Example.** The condition for existence of an optimal measure in Corollary 2B holds for a Gaussian channel  $\Phi$  with the power constraint of the form (19), where  $H = R^T \epsilon R$  is the many-mode oscillator Hamiltonian (see the remark after Proposition 3 in [8]). So, if we assume that  $\bar{\rho}(\mu_*)$  is a Gaussian state, then Corollary 2B shows that  $\bar{C}(\Phi|H, h) = C_{\text{ea}}(\Phi|H, h)$  may be valid only if  $\Phi$  is a c-q channel.

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<sup>5</sup>This means that any spectral projector of the operator  $H$  is diagonalizable in the basis  $\{|k\rangle\}$ .

The above assumption holds provided the conjecture of Gaussian optimizers is valid for the channel  $\Phi$  (see [3, 4] and the references therein).

## Appendix

Let  $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$  be a quantum channel and  $\widehat{\Phi} : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_E)$  be its complementary channel. In finite dimensions the *coherent information* of the channel  $\Phi$  at any state  $\rho$  can be defined as a difference between  $H(\Phi(\rho))$  and  $H(\widehat{\Phi}(\rho))$  [12, 15]. Since in infinite dimensions these values may be infinite even for the state  $\rho$  with finite entropy, for any such state the coherent information can be defined via the quantum mutual information as follows

$$I_c(\Phi, \rho) = I(\Phi, \rho) - H(\rho).$$

Let  $\rho$  be a state in  $\mathfrak{S}(\mathcal{H}_A)$  with finite entropy. By monotonicity of the relative entropy the values  $\chi(\Phi(\mu))$  and  $\chi(\widehat{\Phi}(\mu))$  do not exceed  $H(\rho) = \chi(\mu)$  for any measure  $\mu \in \mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A))$  with the barycenter  $\rho$ . The following lemma can be considered as a generalized version of the observation in [15].

**Lemma 1.** *Let  $\mu$  be a measure in  $\mathcal{P}(\text{extr}\mathfrak{S}(\mathcal{H}_A))$  with the barycenter  $\rho$ . Then*

$$\chi(\Phi(\mu)) - \chi(\widehat{\Phi}(\mu)) = I(\Phi, \rho) - H(\rho) = I_c(\Phi, \rho). \quad (24)$$

This lemma shows, in particular, that the difference  $\chi(\Phi(\mu)) - \chi(\widehat{\Phi}(\mu))$  does not depend on  $\mu$ . So, if the supremum in expression (17) for the value  $\bar{C}(\Phi, \rho)$  is achieved at some measure  $\mu_*$  then the supremum in the similar expression for the value  $\bar{C}(\widehat{\Phi}, \rho)$  is achieved at this measure  $\mu_*$  and vice versa.

**Proof.** If  $H(\Phi(\rho)) < +\infty$  then  $H(\widehat{\Phi}(\rho)) < +\infty$  by the triangle inequality and (24) can be derived from (9) by using the second formula in (16) and by noting that the functions  $\rho \mapsto H(\Phi(\rho))$  and  $\rho \mapsto H(\widehat{\Phi}(\rho))$  coincide on the set of pure states. In general case it is necessary to use the approximation method to prove (24). To realize this method we have to introduce some additional notions.

Let  $\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \mid A \geq 0, \text{Tr} A \leq 1\}$ . We will use the following two extensions of the von Neumann entropy to the set  $\mathfrak{T}_1(\mathcal{H})$  (cf.[11])

$$S(A) = -\text{Tr} A \log A \quad \text{and} \quad H(A) = S(A) + \text{Tr} A \log \text{Tr} A, \quad \forall A \in \mathfrak{T}_1(\mathcal{H}).$$

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy imply the same properties of the functions  $S$  and  $H$  on the set  $\mathfrak{T}_1(\mathcal{H})$ .

The relative entropy for two operators  $A$  and  $B$  in  $\mathfrak{T}_1(\mathcal{H})$  is defined as follows (cf.[11])

$$H(A \| B) = \sum_i \langle i | (A \log A - A \log B + B - A) | i \rangle,$$

where  $\{|i\rangle\}$  is the orthonormal basis of eigenvectors of  $A$ . By means of this extension of the relative entropy the  $\chi$ -quantity of a measure  $\mu$  in  $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$  is defined by expression (15).

A completely positive trace-non-increasing linear map  $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$  is called *quantum operation* [12]. For any quantum operation  $\Phi$  the Stinespring representation (3) holds, in which  $V$  is a contraction. The complementary operation  $\hat{\Phi} : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_E)$  is defined via this representation by (4).

By the obvious modification of the arguments used in the proof of Proposition 1 in [8] one can show that the function  $\mu \mapsto \chi(\mu)$  is lower semicontinuous on the set  $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$  and that for an arbitrary quantum operation  $\Phi$  and a measure  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  such that  $S(\Phi(\bar{\rho}(\mu))) < +\infty$  the  $\chi$ -quantity of the measure  $\Phi(\mu) \doteq \mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$  can be expressed as follows

$$\chi(\Phi(\mu)) = S(\Phi(\bar{\rho}(\mu))) - \int_{\mathfrak{S}(\mathcal{H}_A)} S(\Phi(\rho)) \mu(d\rho). \quad (25)$$

We are now in a position to prove (24) in general case.

Note that for a given measure  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  the function  $\Phi \mapsto \chi(\Phi(\mu))$  is lower semicontinuous on the set of all quantum operations endowed with the strong convergence topology (in which  $\Phi_n \rightarrow \Phi$  means  $\Phi_n(\rho) \rightarrow \Phi(\rho)$  for all  $\rho$  in the trace norm [9]). This follows from lower semicontinuity of the function  $\mu \mapsto \chi(\mu)$  on the set  $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$ , since for an arbitrary sequence  $\{\Phi_n\}$  of quantum operations strongly converging to a quantum operation  $\Phi$  the sequence  $\{\Phi_n(\mu)\} \subset \mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$  weakly converges to the measure  $\Phi(\mu)$  (this can be verified directly by using the definition of the weak convergence and by noting that for sequences of quantum operations the strong convergence is equivalent to the uniform convergence on compact subsets of  $\mathfrak{S}(\mathcal{H}_A)$ ).

Let  $\{P_n\}$  be an increasing sequence of finite rank projectors in  $\mathfrak{B}(\mathcal{H}_B)$  strongly converging to  $I_B$ . Consider the sequence of quantum operations  $\Phi_n = \Pi_n \circ \Phi$ , where  $\Pi_n(\cdot) = P_n(\cdot)P_n$ . Then

$$\hat{\Phi}_n(\rho) = \text{Tr}_{\mathcal{H}_B} P_n \otimes I_{\mathcal{H}_E} V \rho V^*, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad (26)$$

where  $V$  is the isometry from Stinespring representation (3) for the channel  $\Phi$ .

The sequences  $\{\Phi_n\}$  and  $\{\widehat{\Phi}_n\}$  strongly converges to the channels  $\Phi$  and  $\widehat{\Phi}$  correspondingly. Let  $\rho = \sum_k \lambda_k |k\rangle\langle k|$  and  $|\varphi_\rho\rangle = \sum_k \sqrt{\lambda_k} |k\rangle \otimes |k\rangle$ . Since  $S(\Phi_n(\rho)) < +\infty$ , the triangle inequality implies  $S(\widehat{\Phi}_n(\rho)) < +\infty$ . So, we have

$$\begin{aligned} I(\Phi_n, \rho) &= H(\Phi_n \otimes \text{Id}_R(|\varphi_\rho\rangle\langle\varphi_\rho|) \| \Phi_n(\rho) \otimes \varrho) \\ &= -S(\widehat{\Phi}_n(\rho)) + S(\Phi_n(\rho)) + a_n = -\chi(\widehat{\Phi}_n(\mu)) + \chi(\Phi_n(\mu)) + a_n, \end{aligned} \quad (27)$$

where  $a_n = -\sum_k \text{Tr}(\Phi_n(|k\rangle\langle k|)) \lambda_k \log \lambda_k$  and the last equality is obtained by using (25) and coincidence of the functions  $\rho \mapsto S(\Phi(\rho))$  and  $\rho \mapsto S(\widehat{\Phi}(\rho))$  on the set of pure states.

Since the function  $\Phi \mapsto I(\Phi, \rho)$  is lower semicontinuous (by lower semicontinuity of the relative entropy) and  $I(\Phi_n, \rho) \leq I(\Phi, \rho)$  for all  $n$  by monotonicity of the relative entropy under action the quantum operation  $\Pi_n \otimes \text{Id}_R$ , we have

$$\lim_{n \rightarrow +\infty} I(\Phi_n, \rho) = I(\Phi, \rho). \quad (28)$$

We will also show that

$$\lim_{n \rightarrow +\infty} \chi(\Phi_n(\mu)) = \chi(\Phi(\mu)) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \chi(\widehat{\Phi}_n(\mu)) = \chi(\widehat{\Phi}(\mu)). \quad (29)$$

The first relation in (29) follows from lower semicontinuity of the function  $\Phi \mapsto \chi(\Phi(\mu))$  (established before) and the inequality  $\chi(\Phi_n(\mu)) \leq \chi(\Phi(\mu))$  valid for all  $n$  by monotonicity of the  $\chi$ -quantity under action of the quantum operation  $\Pi_n$ .

To prove the second relation in (29) note that (26) implies  $\widehat{\Phi}_n(\rho) \leq \widehat{\Phi}(\rho)$  for any state  $\rho \in \mathfrak{S}(\mathcal{H}_A)$ . Thus Lemma 2 in [9] shows that

$$\chi(\widehat{\Phi}_n(\mu)) \leq \chi(\widehat{\Phi}(\mu)) + f(\text{Tr} \widehat{\Phi}_n(\rho)) \quad (30)$$

where  $f(x) = -2x \log x - (1-x) \log(1-x)$ , for any measure  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  with finite support and the barycenter  $\rho$ . To prove that (30) holds for any measure  $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$  with the barycenter  $\rho$  one can take the sequence  $\{\mu_n\}$  of measures with finite support and the barycenter  $\rho$  constructed in the proof of Lemma 1 in [8], which weakly converges to the measure  $\mu$ , and use lower semicontinuity of the function  $\mu \mapsto \chi(\Psi(\mu))$ , where  $\Psi$  is a quantum

operation, and the inequality  $\chi(\widehat{\Phi}(\mu_n)) \leq \chi(\widehat{\Phi}(\mu))$  valid for all  $n$  by the construction of the sequence  $\{\mu_n\}$  and convexity of the relative entropy.

Inequality (30) and lower semicontinuity of the function  $\Phi \mapsto \chi(\Phi(\mu))$  imply the second relation in (29).

Since  $\{a_n\}$  obviously tends to  $H(\rho)$ , (27), (28) and (29) imply (24).  $\square$

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